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Renewal processes and the Hurst effect

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Abstract

Consider a pure recurrent positive renewal process generated by some inter-arrival waiting time. If the waiting time has power-law fall-off exponent (i.e. tail index) in (1, 2), and if the jump's amplitude has non-zero but finite mean and finite variance, the cumulative amplitude process is long-range dependent with Hurst exponent in (1/2, 1). (Results in this direction have been obtained by Daley under the sole assumptions that the waiting time has moment index in (1, 2)). If the amplitude has zero mean, up to a Brownian trend, the cumulative amplitude process exhibits a *negative-dependence* property with Hurst exponent in (0, 1/2).

The case of delayed stationary renewal processes is also investigated, together with two classes of limiting renewal processes: the compound exponential and the Lévy classes. These are of some interest when the average time between consecutive jumps tends to zero jointly with the probability mass of the jumps' height concentrating about zero in some precise way. Under suitable hypothesis, the Hurst effect is maintained in the limit.

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1. Introduction

The hydrologist Hurst, working on annual flow data of the river Nile, pointed out an empirical phenomenon to be identified later as long-range dependence. This motivated Mandelbrot and coworkers to design the fractional Brownian motion as one of the possible models to explain long memory (see [19] for a landmark and [2] for alternative models). Here, we are concerned with yet another model exhibiting similar statistical features related to Feller's compound renewal processes (or continuous time random walks in the terminology of [14]) with infinite variance for their waiting times but finite variance for amplitude/rewards. Two limiting processes of interest are also investigated from this point of view. These arise when the mean waiting time goes to zero jointly with the probability mass of the jumps' height concentrating about zero in a suitable way. Such limiting constructions of compound renewal processes share the statistical feature that microevents are, in some specific sense to be discussed, extremely frequent.

2. Renewal processes: definitions and notations

We first recall salient facts arising from the modelling of events occurring randomly in time. This is a review of classical material [4], whose purpose is to introduce the definitions and notations used throughout this paper. See also the recent work of [10] and the references therein for similar preoccupations in physics.

2.1. The elementary counting process

Suppose at time $t = 0$, some event occurs for the first time. Suppose successive events occur in the future in such a way that the waiting times between consecutive events form an independent and identically distributed (iid) sequence $(T, T_m, m \geqslant 1)$, with

$$T_m \stackrel{d}{=} T \qquad m \geqslant 1. \tag{1}$$

We shall need the probability distribution function (pdf) of T and its complement to one (cpdf), i.e.

$$F_T(t) := \boldsymbol{P}(T \leqslant t) \qquad \text{and} \qquad \bar{F}_T(t) := 1 - F_T(t). \tag{2}$$

This distribution is assumed to be non-lattice in the sequel.

We are then left with a sequence of events occurring at times

$$\bar{T}(0) = 0 \qquad \bar{T}(n) := \sum_{m=1}^{n} T_m \qquad n \geqslant 1. \tag{3}$$

Let $\bar{N}(t) := \bar{N}([0, t]), t \geqslant 0$, count the random number of events which occurred in the time interval $[0, t]$, with $\bar{N}(0) = 1$. Clearly,

$$\bar{N}(t) = 1 + \sum_{n \geqslant 1} \mathbf{1}(\bar{T}(n) \leqslant t) = \inf(n : \bar{T}(n) > t) \tag{4}$$

with $\mathbf{1}(\cdot)$ the set indicator function which takes the value one if the event is realized, zero, otherwise. As a result, two essential features of such processes are that the events '$\bar{N}(t) > n$' and '$\bar{T}(n) \leqslant t$' coincide, and that $\bar{N}(t)$ satisfies the identity in distribution

$$\bar{N}(t) \stackrel{d}{=} 1 \cdot \mathbf{1}(T > t) + (1 + \bar{N}(t - T)) \cdot \mathbf{1}(T \leqslant t). \tag{5}$$

Besides, it has been known, from the law of large numbers, that, as $t \uparrow \infty$

$$\frac{1}{t}\bar{N}(t) \to \frac{1}{\boldsymbol{E}(T)} \quad \text{a.s.} \tag{6}$$

Such random processes are called *pure* counting renewal processes (the adjective pure is relative to the hypothesis which has been made that the origin of time is an instant at which some event occurred; if this not the case, the adjective *delayed* is currently employed and the first event occurs at time $\bar{T}(0) := T_0 > 0$, independent of $(T, T_m, m \geqslant 1)$ but not necessarily with the same distribution). If in addition $F_T(\infty) = 1$ (T is 'proper') such renewal processes are said to be *recurrent*; this has to be opposed to *transient* renewal processes for which $F_T(\infty) < 1$, corresponding to 'defective' T, allowing for a finite probability that the first event never occurs, i.e. occurs at time $t = +\infty$. In the following, we shall avoid transient processes and limit ourselves to recurrent processes. However, among recurrent processes, we shall distinguish between *positive recurrent* processes for which the average renewal time $\boldsymbol{E}(T) := \theta < +\infty$ and *null recurrent* processes for which $\boldsymbol{E}(T) = +\infty$. In the sequel, we shall avoid null recurrent processes and limit ourselves to positive recurrent ones.

Finally, we note that if the holding times $(T, T_m, m \geqslant 1)$ are exponentially distributed, this counting process boils down into the familiar Poisson process.

2.1.1. The renewal function. We shall call the convergent series

$$U(t) := \boldsymbol{E}\bar{N}(t) = \mathbf{1}(t \geqslant 0) + \sum_{n\geqslant 1} \boldsymbol{P}(\bar{T}(n) \leqslant t) \tag{7}$$

the *intensity* of the pure renewal process. The function $U(t)/t$ is called the *frequency* of the phenomenon. We shall also be interested in the variance function

$$\sigma^2(\bar{N}(t)) = \sum_{n_1, n_2 \geqslant 0} C(B_{n_1}(t), B_{n_2}(t)) \tag{8}$$

where $(B_n(t) := \mathbf{1}(\bar{T}(n) \leqslant t), n \geqslant 1)$ is a sequence of dependent Bernoulli variables with crossed covariances $C(B_{n_1}(t), B_{n_2}(t)), n_1, n_2 \geqslant 0$.

The intensity $U(t)$, also called the renewal function, satisfies the renewal equation [9]

$$U(t) = 1 + \int_{[0,t]} U(t-s)\, \mathrm{d}F_T(s) \tag{9}$$

so that its Laplace transform $U^\wedge(p) := \int_0^\infty \mathrm{e}^{-pt} U(t)\, \mathrm{d}t$ is

$$U^\wedge(p) = \frac{1}{p(1 - \phi_T(p))} \tag{10}$$

with $\phi_T(p) := \int_0^\infty \mathrm{e}^{-pt}\, \mathrm{d}F_T(t)$. From Fatou's and Wald's lemmas, it is also known that, as a corollary to (6)

$$\frac{1}{t} U(t) \to_{t\uparrow\infty} \frac{1}{\theta} \tag{11}$$

holds (the elementary renewal theorem). This result can also be obtained from (10) and the Tauberian theorem [4, 9].

Note also from (10) that the renewal function U can be expressed as

$$U(t) := \mathbf{1}(t \geqslant 0) + U_+(t) \qquad \text{with} \quad U_+(0) = 0.$$

Besides, $U_+^\wedge(p) := \int_0^\infty \mathrm{e}^{-pt} U_+(t)\, \mathrm{d}t = \frac{\phi_T(p)}{p(1-\phi_T(p))}$ and, with $u^\wedge(p) := \int_0^\infty \mathrm{e}^{-pt}\, \mathrm{d}U(t)$ and $u_+^\wedge(p) := \int_0^\infty \mathrm{e}^{-pt}\, \mathrm{d}U_+(t)$, we have $u^\wedge(p) = 1 + u_+^\wedge(p)$ and

$$u^\wedge(p) = \frac{1}{1 - \phi_T(p)} \qquad \text{and} \qquad u_+^\wedge(p) = \frac{\phi_T(p)}{1 - \phi_T(p)}. \tag{12}$$

If $\theta < \infty$, consider the function $\frac{1}{\theta} \int_{[t,\infty)} \bar{F}_T(s)\, \mathrm{d}s$. It is easily checked to be the cpdf of some positive random variable, say A, so that $\bar{F}_A(t) := \frac{1}{\theta} \int_{[t,\infty)} \bar{F}_T(s)\, \mathrm{d}s$, with

$$\theta_A := \boldsymbol{E}A = \frac{\theta}{2} \frac{\boldsymbol{E}(T^2)}{\boldsymbol{E}(T)^2} > \frac{\theta}{2} \tag{13}$$

which is finite if and only if $\boldsymbol{E}(T^2) < \infty$. In this case, this function is decreasing and Riemann integrable hence direct Riemann integrable.

Besides, the Laplace–Stieltjes transform (LST) $\phi_A(p) := \int_0^\infty \mathrm{e}^{-pt}\,\mathrm{d}F_A(t)$ reads

$$\phi_A(p) = \frac{1-\phi_T(p)}{\theta p}. \tag{14}$$

This random variable appears in the following context for positive recurrent renewal processes. Consider the forward recurrence time (FRT) defined by

$$A(t) = \bar{T}(\bar{N}(t)) - t. \tag{15}$$

As is well known, for pure recurrent-positive processes

$$A(t) \underset{t\uparrow\infty}{\overset{d}{\to}} A \qquad \text{with pdf given by} \qquad F_A(t) = \frac{1}{\theta}\int_{[0,t]} \bar{F}_T(s)\,\mathrm{d}s \tag{16}$$

so that A is the limiting forward recurrence time.

Consider now the discrepancy function, which is

$$\tilde{U}(t) := U(t) - t/\theta. \tag{17}$$

From (9), it solves the renewal equation

$$\tilde{U}(t) = \bar{F}_A(t) + \int_{[0,t]} \tilde{U}(t-s)\,\mathrm{d}F_T(s) \tag{18}$$

so that, from the key renewal theorem

$$\tilde{U}(t) = \int_{[0,t]} \bar{F}_A(t-s)\,\mathrm{d}U(s) \tag{19}$$

and (from the direct Riemann integrability of $\bar{F}_A(t)$)

$$\tilde{U}(t) \to_{t\uparrow\infty} \frac{1}{\theta}\int_0^\infty \bar{F}_A(t)\,\mathrm{d}t = \frac{\theta_A}{\theta} > \frac{1}{2}. \tag{20}$$

If $\boldsymbol{E}(T^2) < \infty$, this quantity is finite, whereas if $\boldsymbol{E}(T^2) = \infty$, the discrepancy function diverges and $\tilde{U}(t) \geqslant 0$ (see [5]).

Observing that the LST of $\bar{F}_A(t)$ is $\frac{1}{p}[1-(1-\phi_T(p))/(\theta p)]$, the LST $\tilde{U}^\wedge(p) := \int_0^\infty \mathrm{e}^{-pt}\tilde{U}(t)\,\mathrm{d}t$ reads

$$\tilde{U}^\wedge(p) = \frac{1}{p}\left[1 - \frac{1-\phi_T(p)}{\theta p}\right]\cdot\frac{1}{1-\phi_T(p)} = \frac{\phi_T(p)-(1-\theta p)}{\theta p^2[1-\phi_T(p)]}. \tag{21}$$

With $\tau(q) := \boldsymbol{E}T^q$, the moment generating function of T, assume that the following hypothesis, say (H_δ), holds

$$(H_\delta): \qquad \delta := \sup(q > 0 : \tau(q) < \infty) - 1 \in (0,1) \tag{22}$$

which entails $\tau(2) = \infty$.

A particular case where (H_δ) holds is when

$$(T_\delta): \qquad \bar{F}_T(t) \sim_{t\uparrow\infty} t^{-(1+\delta)}L(t) \tag{23}$$

with $L(t)$ a slowly varying function at infinity. From the Tauberian theorem, this also means (see also theorem 8.1.6 in [3]) that

$$\phi_T(p) \sim_{p\downarrow 0} (1-\theta p) + \frac{\Gamma(2-\delta)}{\delta(1-\delta)}p^{1+\delta}L(1/p).$$

Under (T_δ), T has regularly varying tail with index $1+\delta$ and no moments of order strictly larger than $1+\delta$. Note however that $\tau(1+\delta)$ is either finite or infinite. If it is finite, then $\frac{1}{t}L(t)$ is integrable, therefore $L(t) \to_{t\uparrow\infty} 0$. If $\tau(1+\delta) = \infty$, $L(t)$ either tends to 0 or to ∞, or even has no limit (but only a lim inf and a lim sup).

As a result, from (21) and (23)

$$\tilde{U}^{\wedge}(p) \sim_{p\downarrow 0} \frac{\Gamma(2-\delta)}{\delta(1-\delta)\theta^2} p^{-(2-\delta)} L(1/p) \tag{24}$$

so that, from Tauberian and monotone density theorems [3, 8]

$$\lim_{t\uparrow\infty} \frac{\tilde{U}(t)}{L(t)t^{1-\delta}} = \frac{1}{\theta^2\delta(1-\delta)} \tag{25}$$

which gives the rate at which the discrepancy function tends to zero.

2.1.2. The variance function. Next consider the variance function of $\bar{N}(t)$, say $V(t) := \boldsymbol{E}[\bar{N}(t)^2] - \boldsymbol{E}[\bar{N}(t)]^2$. Split this variance into two terms $V(t) := V_2(t) - V_1(t)$, with $V_2(t) = \boldsymbol{E}[\bar{N}(t)^2] - t^2/\theta^2$ and $V_1(t) = \tilde{U}(t)^2 + 2\frac{t}{\theta}\tilde{U}(t)$. Concerning the second term, we easily get

$$\lim_{t\uparrow\infty} \frac{V_1(t)}{L(t)t^{2-\delta}} = \frac{2}{\theta^3\delta(1-\delta)}. \tag{26}$$

Concerning the first term, letting the second moment $M_2(t) := \boldsymbol{E}[\bar{N}(t)^2]$ and $M_2^{\wedge}(p)$ its LST, we arrive at

$$V_2^{\wedge}(p) = M_2^{\wedge}(p) - \frac{2}{\theta^2 p^3}. \tag{27}$$

Now, $M_2(t)$ satisfies the integral equation

$$M_2(t) = 2U(t) - 1 + \int_{[0,t]} M_2(t-s)\,\mathrm{d}F_T(s) \tag{28}$$

so that

$$M_2^{\wedge}(p) = 2U^{\wedge}(p) - \frac{1}{p} + M_2^{\wedge}(p)\phi_T(p). \tag{29}$$

In other words, after some algebraic manipulations

$$M_2^{\wedge}(p) = \frac{1+\phi_T(p)}{p(1-\phi_T(p))^2}. \tag{30}$$

Finally,

$$V_2^{\wedge}(p) \sim_{p\downarrow 0} \frac{4\Gamma(2-\delta)}{\theta^3\delta(1-\delta)} p^{-(3-\delta)} L(1/p) \tag{31}$$

showing that

$$\lim_{t\uparrow\infty} \frac{V_2(t)}{L(t)t^{2-\delta}} = \frac{4}{\theta^3\delta(1-\delta)(2-\delta)}. \tag{32}$$

Finally, we obtain

$$\lim_{t\uparrow\infty} \frac{V(t)}{L(t)t^{2-\delta}} = \frac{2}{\theta^3(1-\delta)(2-\delta)}. \tag{33}$$

Let $H = (2-\delta)/2 \in (1/2, 1)$ be the Hurst exponent of $\bar{N}(t)$. With $\sigma^2(\bar{N}(t)) := V(t)$ the variance function of $\bar{N}(t)$, we arrive at the following conclusion, which, to a large extent, has already been known since Feller:

Theorem 1. *Suppose $\boldsymbol{E}(T) = \theta < \infty$ and $\sigma^2(T) = \infty$. Suppose that T has regularly varying tails with index in* $(1, 2)$.

Then, with $H := \frac{1}{2}\{3 - \sup(q > 0 : \tau(q) < \infty)\} \in (1/2, 1)$, $\bar{N}(t)$ *is long-range dependent with Hurst exponent* H, *in the sense that*

$$\lim_{t\uparrow\infty} \frac{\sigma^2(\bar{N}(t))}{L(t)t^{2H}} = \frac{1}{\theta^3(2H-1)H} \tag{34}$$

which implies $\sigma^2(\bar{N}(t)) = o(t^{2H+\varepsilon})$, *for any* $\varepsilon > 0$, *and even* $\sigma^2(\bar{N}(t)) = o(t^{2H})$ *if* $L(t) \to 0$ $(t \uparrow \infty)$.

Comment. From the point of view of theory, there is some ambiguity there which is worth being underlined. Following [2, 13], one may say that the Hurst effect with exponent $H \in (1/2, 1)$ holds for $\bar{N}(t)$ if, with $R(\bar{N}(t))$ the adjusted range of $\bar{N}(t)$, the quantity $t^{-H}(R(\bar{N}(t))/\sigma(\bar{N}(t)))$ converges in distribution to some random variable which is not 0. This has not been proved. Rather, our definition in (34) stick to the one of Daley's paper [6] which is of a slightly different nature and on which we shall now say a few words.

2.1.3. Daley's approach. It turns out that condition (H_δ) can be realized without T having regularly varying tails (see the example on p 2039 in [6]). In this context, the following stronger result then holds

Theorem 2 [6]**.** *Suppose* $\boldsymbol{E}(T) = \theta < \infty$ *and* $\sigma^2(T) = \infty$.

If $H := \frac{1}{2}\{3 - \sup(q > 0 : \tau(q) < \infty)\} \in (1/2, 1)$, *then* $\bar{N}(t)$ *is long-range dependent with Hurst exponent* H, *in the sense that*

$$\limsup_{t\uparrow\infty} \frac{\sigma^2(\bar{N}(t))}{t} = \infty \tag{35}$$

and

$$H = \inf\left(h > 0 : \limsup_{t\uparrow\infty} \frac{\sigma^2(\bar{N}(t))}{t^{2h}} < \infty\right). \tag{36}$$

In a recent unpublished paper [7], Daley subsequently proved (among other things and without any appeal to Laplace transforms methods) the following results generalizing (25), (33).

Theorem 3 [7]**.** *Under the sole finite mean and infinite variance assumptions on T's distribution*

$$\lim_{t\uparrow\infty} \frac{\tilde{U}(t)}{\int_0^t \mathrm{d}\tau \int_\tau^\infty \bar{F}_T(s)\,\mathrm{d}s} = \frac{1}{\theta^2} \tag{37}$$

$$\lim_{t\uparrow\infty} \frac{V(t)}{\int_0^t s(t-s)\bar{F}_T(s)\,\mathrm{d}s} = \frac{2}{\theta^3}. \tag{38}$$

These results may indeed be checked to give (25), (33) if T's distribution is non-lattice and has regularly varying tails as in (T_δ). Besides, the exact shape of T's (and A's) tail is involved. This is the machinery to be used, with the conclusion drawn in theorem 2, when, as in this author's example mentioned above, T has infinite second moment, without having regularly varying tails. It would be interesting to have some additional insight on this class of models for T to grasp what is missed working with (T_δ) rather than (H_δ). In the following however, we shall limit ourselves to related questions under the restricted hypothesis (T_δ), chiefly because our approach is essentially one using transform methods. By doing so, we probably miss subtleties related to Daley's refinements and there is some open work to be achieved here.

2.2. *Compound recurrent renewal processes and the cumulative magnitude*

The process $\bar{N}(t)$ counts the number of events which occurred before time t. Assume now some physical phenomenon to be described by a compound renewal process: events of random iid amplitudes, say $(X, X_m, m \geqslant 1)$, occur at random times $\bar{T}(n)$, $n \geqslant 1$, the waiting times of which forming an iid sequence. In the physics literature this model is called the continuous-time-random-walk (CTRW).

In the sequel, we shall note F_X and $\bar{F}_X$ the pdf and cpdf of the local amplitude X. One may be interested by a process which cumulates this random number of random amplitudes. A compound renewal process is to the counting renewal process what a compound Poisson process is to a Poisson process itself.

Physical situations where the relevance of this model holds are numerous: think of the random magnitudes as a claims sequence in insurance risk theory, as the energy release of individual earthquakes in geophysics or as random water inputs flowing into a dam in hydrology. Summing the individual contributions yields the total claim amount (cumulative energy release and water input) over a certain lapse of time. In all these applications we have in mind, the magnitude X is a positive random variable; we shall therefore mainly deal with this case in the following (unless otherwise specified).

If the random amplitude X is not positive, one rather speaks of it as of reward.

Let then the cumulative magnitude $\bar{X}(t)$ be defined by

$$\bar{X}(t) = \sum_{m=0}^{\bar{N}(t)} X_m = \sum_{n\geqslant 0} X_n \mathbf{1}(\bar{T}(n) \leqslant t) \tag{39}$$

where $(X, X_m, m \geqslant 0)$ is the iid random amplitude sequence. The distribution of this variable may be derived by using the following identity in distribution:

$$\bar{X}(t) \stackrel{d}{=} X_0 \cdot \mathbf{1}(T > t) + (X(T) + \bar{X}(t - T)) \cdot \mathbf{1}(T \leqslant t) \tag{40}$$

where $T > 0$ is a 'proper' positive random variable known as the first renewal time of $\bar{X}(t)$. Such processes are called compound pure recurrent renewal processes.

Let us briefly comment on this formula. At time T, $\bar{X}(t)$ undergoes a first (random) jump with amplitude $X(T) > 0$, possibly dependent on the occurrence time T of this jump.

Let us now freeze the time t at which $\bar{X}(t)$ is to be evaluated. If the realization of time T exceeds the time t of interest, the process $\bar{X}(t)$ is in the initial state $X_0 \stackrel{d}{=} X$. If $T = s \leqslant t$, the value of $\bar{X}(t)$ is the independent sum of the first jump of amplitude $X(s)$ plus a statistical copy of the process $\bar{X}(\cdot)$ in the remaining time $t - s$, conditionally to the event $T = s$. Renewal processes generalize the familiar compound Poisson process family in that the waiting time distributions between spikes are an iid sequence, albeit not necessarily exponentially distributed.

Suppose that $X \geqslant 0$. Let us now translate definition (40) in terms of the evolution of the Laplace transform of $\bar{X}(t)$. Let

$$\Phi_{\bar{X}}(t, \lambda) := \boldsymbol{E}\, \mathrm{e}^{-\lambda \bar{X}(t)} \qquad \text{and} \qquad \phi_X(s, \lambda) := \boldsymbol{E}\, \mathrm{e}^{-\lambda X(s)} \tag{41}$$

respectively stand for the Laplace transforms of the cumulative process $\bar{X}(t)$ and of a local magnitude $X(s)$ which occurred at time $s \leqslant t$. Then

$$\Phi_{\bar{X}}(t, \lambda) = \phi_X(\lambda) \boldsymbol{P}(T > t) + \int_{[0,t]} \Phi_{\bar{X}}(t - s, \lambda) \phi_X(s, \lambda)\, \mathrm{d}F_T(s). \tag{42}$$

We shall now make an additional simplifying hypothesis.

Assume that the local magnitudes are independent of their occurrence time (the *decoupling* hypothesis); then $\phi_X(s, \lambda) = \phi_X(\lambda)$ and the Laplace transform of the conditional magnitude

X is independent of the particular realization s of the occurrence time T. Then (42) boils down to

$$\Phi_{\bar{X}}(t,\lambda) = \phi_X(\lambda)\boldsymbol{P}(T > t) + \phi_X(\lambda)\int_{[0,t]} \Phi_{\bar{X}}(t-s,\lambda)\,\mathrm{d}F_T(s). \tag{43}$$

This is the integral (convolution) equation that $\Phi_{\bar{X}}(t,\lambda)$ now satisfies. From (12), using the convolution equation (43), the Laplace transform of $\Phi_{\bar{X}}(.,\lambda)$ satisfies

$$\Phi^{\wedge}_{\bar{X}}(p,\lambda) = \frac{(1-\phi_T(p))\phi_X(\lambda)}{p(1-\phi_T(p)\phi_X(\lambda))} = \frac{\phi_X(\lambda)}{p(1+u^{\wedge}_{+}(p)(1-\phi_X(\lambda)))} \tag{44}$$

provided that $\phi_T(p)\phi_X(\lambda) < 1$. Thus, the solutions of $\phi_T(p) = \phi_X(\lambda)^{-1}$ are the poles of $\Phi^{\wedge}_{\bar{X}}(p,\lambda)$.

Processes obeying equation (44) are known as pure *renewal* processes with stationary local magnitudes [9]. This equation is the celebrated Montroll–Weiss equation [14] which led to many further interesting developments in physics [22, 16, 21, 11].

Remark 1. Compound renewal processes are more general than standard processes with stationary independent increments (sii), such as Poisson's, because they are not Markovian as the integral equation (42) shows: the distribution at time t of $\bar{X}(t)$ depends (in general) on the distribution of $\bar{X}(s)$, $s < t$. However, it can easily be shown that they include the compound Poisson class (a very important sub-class of processes with sii) which may be recovered if the renewal time T is assumed to be exponentially distributed, because of the memory-less character of the exponential distribution.

Example 1. From (15), consider the process

$$\bar{A}(t) := t + A(t) = \bar{T}(\bar{N}(t)).$$

Clearly,

$$A(t) \stackrel{d}{=} (T-t)\cdot\mathbf{1}(T > t) + A(t-T)\cdot\mathbf{1}(T \leqslant t)$$

so that

$$\bar{A}(t) \stackrel{d}{=} T\cdot\mathbf{1}(T > t) + (T + \bar{A}(t-T))\cdot\mathbf{1}(T \leqslant t).$$

This process is a compound renewal process whose jump's height is T itself and, as such, is not typical of (44). Its LST functional reads

$$\Phi^{\wedge}_{\bar{A}}(p,\lambda) = \frac{\phi_T(\lambda) - \phi_T(p+\lambda)}{p(1-\phi_T(p+\lambda))}.$$

Note that if T has infinite variance, so does the jump's height of $\bar{A}(t)$.

Let us turn back to our problem.

If $0 < \boldsymbol{E}(X) := \mu < \infty$, we easily conclude that $\boldsymbol{E}(\bar{X}(t)) = \mu U(t)$, hence that

$$\frac{1}{t}\bar{X}(t) \to_{t\uparrow\infty} \frac{\mu}{\theta} \quad \text{a.s.} \qquad \text{and} \qquad \frac{1}{t}\boldsymbol{E}(\bar{X}(t)) \to_{t\uparrow\infty} \frac{\mu}{\theta}. \tag{45}$$

Besides, under (T_δ),

$$\lim_{t\uparrow\infty} \frac{\boldsymbol{E}(\bar{X}(t)) - \mu t/\theta}{t^{1-\delta}} = \frac{\mu}{\theta^2\delta(1-\delta)}. \tag{46}$$

Following the previous steps, we obtain:

Corollary 4. *If $\sigma^2(X) < \infty$, the variance function of $\bar{X}(t)$ is well-defined, and, under (T_δ), with $H = (2-\delta)/2 \in (1/2, 1)$*

$$\lim_{t\uparrow\infty} \frac{\sigma^2(\bar{X}(t))}{L(t)t^{2H}} = \frac{\mu^2}{\theta^3(2H-1)H} \tag{47}$$

generalizing (33).

In this case, $\bar{X}(t)$ is also long-range dependent, with Hurst exponent $H \in (1/2, 1)$. Such long-range dependent processes are super-diffusive or persistent as $H > 1/2$ (as a model of enhanced diffusion).

2.3. The renewal random walk

Here, we consider the particular case where the distribution of X has support $\boldsymbol{R}$, with $\boldsymbol{E}X := \mu = 0$ and variance $\sigma^2(X) < \infty$.

Let $M_1(t) := \boldsymbol{E}(\bar{X}(t))$ and $\sigma^2(t) := \sigma^2(\bar{X}(t))$ stand for the first moment and variance of $\bar{X}(t)$. One may easily check that

$$M_1(t) = \int_{[0,t]} M_1(t-s)\,\mathrm{d}F_T(s) \tag{48}$$

showing that $M_1(t) = 0$, for all $t \geqslant 0$, and

$$\sigma^2(t) = \sigma^2(X) + \int_{[0,t]} \sigma^2(t-s)\,\mathrm{d}F_T(s) \tag{49}$$

showing that $\sigma^2(t) = \sigma^2(X)U(t)$. From the elementary renewal theorem $\sigma^2(t) \to_{t\uparrow\infty} \infty$ and $\sigma^2(t)/t \to_{t\uparrow\infty} \sigma^2(X)/\theta$; this is the standard diffusive regime of a standard random walk, but this regime is attained asymptotically only.

If $\sigma^2(T) < \infty$, with $\tilde{\sigma}^2(t) := \sigma^2(t) - t\sigma^2(X)/\theta$, the variance's discrepancy, we easily get

$$\tilde{\sigma}^2(t) \to_{t\uparrow\infty} \frac{\sigma^2(X)}{2}\left(1 + \frac{\sigma^2(T)}{\theta^2}\right). \tag{50}$$

If $\sigma^2(T) = \infty$ and if (T_δ) holds, with $\delta := \sup(q > 0 : \tau(q) < \infty) - 1 \in (0, 1)$, we have

$$\lim_{t\uparrow\infty} \frac{\sigma^2(\bar{X}(t)) - t\sigma^2(X)/\theta}{L(t)t^{1-\delta}} = \frac{\sigma^2(X)}{\theta^2\delta(1-\delta)} \tag{51}$$

giving the speed at which $\sigma^2(\bar{X}(t))/t$ tends to $\sigma^2(X)/\theta$. In other words, we obtain the following result under (T_δ)

Corollary 5. *Consider the process $\tilde{X}(t)$, $t \geqslant 0$ defined by $\bar{X}(t) \stackrel{d}{=} \tilde{X}(t) + B(t)$, $t \geqslant 0$, where $B(t)$ is a Brownian trend without drift and variance $\sigma^2(B(t)) = t\sigma^2(X)/\theta$, independent of $\tilde{X}(t)$. We have*

$$\boldsymbol{E}(\bar{X}(t)) = \boldsymbol{E}(\tilde{X}(t)) = 0 \qquad \textit{and} \qquad \sigma^2(\bar{X}(t)) = \sigma^2(\tilde{X}(t)) + t\sigma^2(X)/\theta.$$

With $H := (1-\delta)/2 \in (0, 1/2)$, the process $\tilde{X}(t)$ exhibits the negative-dependence property

$$\lim_{t\uparrow\infty} \frac{\sigma^2(\tilde{X}(t))}{L(t)t^{2H}} = \frac{\sigma^2(X)}{2\theta^2 H(1-2H)}. \tag{52}$$

Here $H \in (0, 1/2)$, meaning that $\tilde{X}(t)$ (obtained from $\bar{X}(t)$ while detrending) is sub-diffusive or antipersistent. This behaviour should be compared with that in (47).

Remark 2. (i) Suppose the distribution of X is $\frac{1}{2}\delta(x-\varepsilon)+\frac{1}{2}\delta(x+\varepsilon)$, with mean $\boldsymbol{E}X=0$ and variance $\sigma^2(X)=\varepsilon^2$. From (44), as ε and $\theta\downarrow 0$, while $\varepsilon^2/\theta=D$ constant, $\bar{X}(t)\overset{d}{\to}B(t), t\geqslant 0$, with $\sigma^2(B(t))=Dt$. In this limit, there is no Hurst effect to be expected (see however section 3 for appropriate limits where the Hurst effect is maintained).

(ii) If $\sigma^2(X)=\infty$ then $\bar{X}(t)$ does not even have finite variance (see [17, 18] for ways to handle this problem). As an illustration, this phenomenon occurs for the process $\bar{A}(t)$ defined in example 1, with $\sigma^2(T)=\infty$.

2.4. *Delayed renewal processes and stationarity*

In this section, we investigate the important question of *stationarity* of renewal processes, under the hypothesis that some initial time delay is present.

Consider now a *delayed* process for which the origin of time is *not* an instant at which some event occurred; rather, one has to wait a random time T_0, before the first occurrence of an event is seen and before the mechanics of a pure renewal process proceeds.

Time T_0 is assumed independent of the iid waiting times $(T, T_m, m\geqslant 1)$, but not with the same distribution. Under this hypothesis, the distribution of the delayed process, say $\bar{X}^d(t)$, may be obtained through

$$\bar{X}^d(t)\overset{d}{=}0\cdot\mathbf{1}(T_0>t)+\bar{X}(t-T_0)\cdot\mathbf{1}(T_0\leqslant t)$$

in terms of the shifted underlying pure process $\bar{X}(t)$. In terms of the LST functional, this translates into

$$\Phi^{\wedge}_{\bar{X}^d}(p,\lambda)=\frac{1}{p}\big(1-\phi_{T_0}(p)\big)+\phi_{T_0}(p)\frac{\phi_X(\lambda)}{p(1+u^{\wedge}_{+}(p)(1-\phi_X(\lambda)))}. \tag{53}$$

The distribution of the forward recurrence time $A^d(t)$ of this delayed process $\bar{X}^d(t)$ may be expressed as

$$A^d(t)\overset{d}{=}(T_0-t)\cdot\mathbf{1}(T_0>t)+A(t-T_0)\cdot\mathbf{1}(T_0\leqslant t) \tag{54}$$

in terms of the shifted forward recurrence time $A(t)$ of the underlying pure renewal process. From this observation, it is easy to derive the following important fact:

If the distribution of T_0 is that of the asymptotic forward recurrence time A, then $A^d(t)\overset{d}{=}A$, for all times $t>0$. In other words, the distribution of the forward recurrence for the delayed process, with $T_0\overset{d}{=}A$, is invariant with time. Such delayed renewal processes are called stationary.

The sequence $(\bar{T}(n), n\geqslant 0)$ with $\bar{T}(0)=T_0\overset{d}{=}A$, $\bar{T}(n)=\sum_{m=1}^{n}T_m, n\geqslant 1$ is called a stationary renewal sequence.

If $U^d(t)$ is the renewal function of such stationary processes, clearly $U^d(t)=\int_0^\infty F_{T_0}(ds)U(t-s)$. From $T_0\overset{d}{=}A$ and (10), (14): $U^d(t)=t/\theta$ for all times $t\geqslant 0$. Besides, with $\mu=\boldsymbol{E}(X)$ we have $\boldsymbol{E}(\bar{X}^d(t))=\mu U^d(t)=\mu t/\theta$, for all times. The variance function therefore is $\sigma^2(\bar{X}^d(t))=\boldsymbol{E}(\bar{X}^d(t))^2-[\mu t/\theta]^2$ and, up to minor corrections, we obtain similarly

Corollary 6. *Assume $\sigma^2(X)<\infty$. Under (T_δ), the long-range dependence behaviour (47) also holds for the stationary delayed process and, with $H=(2-\delta)/2\in(1/2,1)$*

$$\lim_{t\uparrow\infty}\frac{\sigma^2(\bar{X}^d(t))}{L(t)t^{2H}}=\frac{\mu^2}{\theta^3(2H-1)H(H-1)}. \tag{55}$$

3. Two classes of limiting stationary renewal processes: connections with the theory of subordinators

In this section, we proceed with the construction of two limiting compound renewal processes. By doing so, we exhibit some connections which exist between renewal processes and the theory of subordinators [1].

Our physical motivations are the following: in many domains of applications, the spacetime phenomenon to be studied presents itself as a sequence of events with random amplitude occurring randomly in time. To take a physical image, think of it as a sequence of earthquake magnitudes over some region or the sequence of damages met by the customers of some insurance company in finance (see [12, 8], for precise motivations and results in both physics and finance).

So far, the following simple statistical model for such sequences was considered: events of random iid amplitude, say $(X; X_n, n \geqslant 0)$, occur randomly in time according to a delayed renewal process, say $\bar{N}^d(t)$, with intensity $\theta t, m > 0$. This process, say $X^d(t), t \geqslant 0$, is a continuous-time process called the jump (or amplitude) process defined by: $X^d(t) = X_{\bar{N}(t)}$ if $\bar{N}(t) > 0$, $X^d(t) = 0$ otherwise.

Upon cumulating these magnitudes over time, we are left with a simple compound *renewal* process $\bar{X}^d(t)$ which integrates the previous jump process.

In many of the examples of interest discussed above, an additional characteristic feature of the phenomenon under study is the following: events of tiny amplitude (say larger than but close to some threshold $\varepsilon > 0$, with ε small) are extremely numerous or frequent. Thus, there exists some $\varepsilon > 0$ which serves as the left-endpoint of the support of magnitude X's distribution.

To stick to our physical image, an important feature of earthquake catalogues is that, although data on small earthquakes are strongly deficient, due to incomplete, bad registration of low magnitude events, these are certainly extremely numerous or frequent. One possible way to address this problem is to consider the data as a realization of a truncated distribution; in effect, this amounts to the assumption that there exists some fixed detection threshold ε above which all earthquakes are recorded. Such a threshold could have been either estimated from the data or deduced from physical considerations, at the price of discarding part of the data. Lowering this threshold, a huge number of small events will certainly emerge.

On the other hand, these are punctuated with some rare events but with comparatively very large macroscopic amplitude (the ones of interest to the engineer): then, the physical image becomes the one of bursts of activity immersed in an ocean of 'microevents' (a coarse version of Lévy noise). Our goal here is to study two limiting constructions of compound renewal processes which share the statistical feature of interest that microevents are in some sense to be discussed 'extremely frequent'.

Before we proceed, let us start with elementary algebra on stationary compound renewal processes, completing the previous section.

For a stationary delayed renewal process, putting $\phi_{T_0}(p) = \frac{1}{p\theta}(1 - \phi_T(p))$ into (53), we arrive at

$$\Phi^{\wedge}_{\bar{X}^d}(p, \lambda) = \frac{1}{p}\left[1 - \frac{1 - \phi_X(\lambda)}{p\theta(1 + u^{\wedge}_{+}(p)(1 - \phi_X(\lambda)))}\right]. \tag{56}$$

Note that $u^{\wedge}_{+}(p)(1 - \phi_X(\lambda)) = \int_{(0,\infty)^2} \mathrm{e}^{-pt}(1 - \mathrm{e}^{-\lambda x})\, \mathrm{d}U_+(t) \otimes \mathrm{d}F_X(x)$, where $F_X(x)$ is the pdf of the jumps' height of the stationary compound renewal process $\bar{X}^d(\cdot)$. The product measure is called the local characteristics of $\bar{X}^d(\cdot)$. If T's law is $\mathrm{Expo}\left(\frac{1}{\theta}\right)$,

$u_+^\wedge(p) = 1/(p\theta)$ ($\mathrm{d}U_+(t) = \frac{1}{\theta}\,\mathrm{d}t$ is the Lebesgue measure) and

$$\Phi^\wedge_{\bar{X}^d}(p, \lambda) = \frac{1}{p}\left[1 - \frac{1-\phi_X(\lambda)}{p\theta + (1-\phi_X(\lambda))}\right] = \frac{1}{p + (1-\phi_X(\lambda))/\theta}$$

from which we find $\Phi_{\bar{X}^d(t)}(\lambda) = \exp[-t/\theta(1-\phi_X(\lambda))]$ which is the LST of a compound Poisson process, as required.

3.1. Compound exponential stationary renewal processes

Compound exponential stationary renewal processes, now defined, will constitute a particular class of limiting compound stationary renewal processes. We now proceed first with such construction.

Let $T^1 \geqslant 0$ be the proper waiting time of some pure renewal process. Assume $\boldsymbol{E}(T^1) = 1$. Let $V(t)$, $\mathrm{d}V(t)$ and $\mathrm{d}V_+(t)$ be the associated renewal function and measures. Let $\theta \in (0, 1)$. Consider the waiting time $T = B \cdot T^1$ with B a Bernoulli random variable $\boldsymbol{P}(B = 1) = \theta$, independent of T^1. Clearly $\boldsymbol{E}(T) = \theta$ and the renewal process generated by such T has for renewal function $U(t) = \frac{1}{\theta}V(t)$. Besides, $\mathrm{d}U = \frac{1}{\theta}\,\mathrm{d}V$.

Note that $\mathrm{d}U_+ = \frac{1}{\theta}[\mathrm{d}V - \theta\delta_0\,\mathrm{d}t]$ so that $\mathrm{d}U_+ \neq \frac{1}{\theta}\,\mathrm{d}V_+$.

In terms of LST: $U^\wedge(p) = \frac{1}{\theta}V^\wedge(p)$, $u^\wedge(p) = \frac{1}{\theta}v^\wedge(p)$ and

$$u_+^\wedge(p) = \frac{1}{\theta}[v^\wedge(p) - \theta]. \tag{57}$$

Note that $u_+^\wedge(p) = \frac{1}{\theta}[v_+^\wedge(p) + 1 - \theta] \neq \frac{1}{\theta}v_+^\wedge(p)$.

Let $\Pi_Z(\mathrm{d}x)$ be a positive σ-finite Lévy measure on $(0, \infty)$ with total mass ∞. Assume that $\int_0^\infty (1 \wedge x)\Pi_Z(\mathrm{d}x) < \infty$. Let also $\psi(\lambda) := \int_0^\infty (1 - \mathrm{e}^{-\lambda x})\Pi_Z(\mathrm{d}x)$, with $\psi(0) = 0$, $\psi(\infty) = \infty$.

It is known that $\Phi_{\bar{Z}(t)}(\lambda) = \mathrm{e}^{-t\psi(\lambda)}$ is the LST of some Lévy subordinator $\bar{Z}(t)$, $t \geqslant 0$ and ψ is known as its Laplace exponent. Lévy subordinators constitute an important extension of the class of compound Poisson processes [1].

Lemma 7. *Let ψ be such a Laplace exponent. The function $\phi_Z(\lambda) = 1/(1 + \psi(\lambda))$ is the LST of some infinitely divisible random variable, say Z.*

Proof. Let E be some exponential random variable with mean 1, independent of $\bar{Z}(t)$, $t \geqslant 0$. Consider the random variable $\bar{Z}(E)$. We have $\boldsymbol{E}\,\mathrm{e}^{-\lambda\bar{Z}(E)} = 1/(1 + \psi(\lambda))$. This standard claim results from the infinite divisibility of E and of $\bar{Z}(1)$. □

The class of infinitely divisible variables defined in this way is called the compound exponential class.

Next consider the average time spent by $\bar{Z}(t)$ below $x > 0$ and set

$$\mathcal{U}_+(x) := \boldsymbol{E}\int_0^\infty \mathbf{1}(\bar{Z}(t) \leqslant x)\,\mathrm{d}t. \tag{58}$$

This function is known as the potential function of $\bar{Z}(\cdot)$.

Corollary 8. *This potential function admits the alternative representation*

$$\mathcal{U}_+(x) = \boldsymbol{E}\left(\sum_{n\geqslant 1}\mathbf{1}(\bar{Z}(n) \leqslant x)\right) \tag{59}$$

where $\bar{Z}(n) := \sum_{m=1}^{n} Z_m$*, with* $(Z; Z_m, m \geqslant 1)$ *is some iid sequence characterized by the LST:* $\phi_Z(\lambda) := \boldsymbol{E}\,\mathrm{e}^{-\lambda Z} = 1/(1+\psi(\lambda))$.

Proof. The LST, say $u_+^{\wedge}(\lambda)$, of the potential measure associated to $\mathcal{U}$ in (58) is easily shown to be

$$u_+^{\wedge}(\lambda) = \frac{1}{\psi(\lambda)} \qquad \lambda > 0 \tag{60}$$

which, for some LST ϕ_Z, is also

$$u_+^{\wedge}(\lambda) = \frac{1}{\psi(\lambda)} = \frac{\phi_Z(\lambda)}{1-\phi_Z(\lambda)}.$$

From (7), (12), this means that (59) holds. □

In (56), suppose the support of X is $[\varepsilon, \infty]$, with $\varepsilon > 0$. Call this random variable X_ε. With $\bar{\Pi}_Z(x) := \Pi_Z[x, \infty)$, the tail of the Lévy measure, suppose the cpdf of X_ε reads $\bar{F}_{X_\varepsilon}(x) = \bar{\Pi}_Z(x \vee \varepsilon)/\bar{\Pi}_Z(\varepsilon)$. Suppose also that θ depends on ε; call it θ_ε. Call such compound renewal processes $\bar{X}_\varepsilon^d(t)$.

We shall now pass to the limit $\varepsilon \downarrow 0$, while $\theta_\varepsilon \bar{\Pi}_Z(\varepsilon) = 1$ in (56). Note that in this process $\bar{\Pi}_Z(\varepsilon) \to \infty$ and that $\theta_\varepsilon \to 0$ (the average time between consecutive jumps tends to 0, while the probability mass of the jumps X_ε concentrates at 0). It turns out that $\bar{X}_\varepsilon^d(\cdot)$ converges weakly to some limiting process $\bar{X}_0^d(\cdot)$ which we shall call a compound exponential stationary renewal processes. Indeed, we obtain

Theorem 9. *The following weak convergence holds*

$$\left(\bar{X}_\varepsilon^d(t), t \geqslant 0\right) \xrightarrow{d} \left(\bar{X}_0^d(t), t \geqslant 0\right) \qquad \varepsilon \downarrow 0$$

where $\bar{X}_0^d(\cdot)$ *is a compound renewal process with infinitely divisible jump's height Z and renewal function* V_+.

Proof. From this construction, observing that $\frac{1}{\theta_\varepsilon}(1-\phi_{X_\varepsilon}(\lambda)) \to_{\varepsilon\downarrow 0} \psi(\lambda)$, and from (57), we obtain the LST functional of this stationary renewal processes as

$$\Phi_{\bar{X}_0^d}^{\wedge}(p,\lambda) = \frac{1}{p}\left[1 - \frac{\psi(\lambda)}{p\,(1+v^{\wedge}(p)\psi(\lambda))}\right]. \tag{61}$$

Compound exponential stationary renewal processes for which $\psi^{(1)}(0) < \infty$ (respectively $\psi^{(2)}(0) < \infty$) are such that $\bar{X}^d(t)$ has finite mean (respectively variance) for all $t \geqslant 0$. Let $\phi_Z(\lambda) := 1/(1+\psi(\lambda))$. This is the LST of some positive random variable with pdf, say F_Z. From this, and recalling that $v^{\wedge}(p) = 1 + v_+^{\wedge}(p)$, formula (61) can be written as

$$\Phi_{\bar{X}_0^d}^{\wedge}(p,\lambda) = \frac{1}{p}\left[1 - \frac{1-\phi_Z(\lambda)}{p(1+v_+^{\wedge}(p)(1-\phi_Z(\lambda)))}\right] \tag{62}$$

which is of the type (56). Our weak convergence claim follows from the continuity theorem for sequences of characteristic functions, applying the analogous theorem for sequences of LSTs in the temporal domain [9].

Note that $v_+^{\wedge}(p)(1-\phi_Z(\lambda)) = \int_{(0,\infty)^2} \mathrm{e}^{-pt}(1-\mathrm{e}^{-\lambda x})\,\mathrm{d}V_+(t) \otimes \mathrm{d}F_Z(x)$, where $F_Z(x)$ is the pdf of the jumps' height of the limiting stationary compound renewal process $\bar{X}_0^d(\cdot)$.

Note from (61) that if T^1 has Expo(1) distribution, $v^{\wedge}(p) = 1/(1+p)$ and $\Phi_{\bar{X}^d}^{\wedge}(p,\lambda) = (p + \psi(\lambda)/(1+\psi(\lambda)))^{-1}$ from which we get

$$\Phi_{\bar{X}^d}(t,\lambda) = \mathrm{e}^{-t[1-1/(1+\psi(\lambda))]}.$$

This is the LST of a compound Poisson subordinator with sii, whose jump height has LST: $\phi_Z(\lambda) := 1/(1+\psi(\lambda))$. □

Let A^1 be the limiting FRT of the pure renewal process generated by T^1. The sequence $(\bar{T}^1(n), n \geqslant 0)$ with $\bar{T}^1(0) := T_0^1 \overset{d}{=} A^1$, $\bar{T}^1(n) = \sum_{m=1}^n T_m^1, n \geqslant 1$ is a stationary renewal sequence.

If $V^d(t)$ is the renewal function of such stationary processes, clearly $V^d(t) = \int_0^\infty F_{T_0^1}(\mathrm{d}s)V(t-s)$. From $T_0^1 \overset{d}{=} A^1$ and (10), (14): $V^d(t) = t$ for all times $t \geqslant 0$. Besides, if $\mu = \boldsymbol{E}(Z) = \psi^{(1)}(0) < \infty$ we have $\boldsymbol{E}\big(\bar{X}_0^d(t)\big) = \mu V^d(t) = \mu t$, for all times. The variance function therefore is $\sigma^2\big(\bar{X}_0^d(t)\big) = \boldsymbol{E}\big(\bar{X}_0^d(t)\big)^2 - (\mu t)^2$ and, from (55), (56), we obtain similarly

Corollary 10. *Assume $\sigma^2(Z) < \infty$. Under (T_δ) for T^1, the long-range dependence behaviour (47) also holds for the limiting compound exponential delayed process and, with $H = (2-\delta)/2 \in (1/2, 1)$*

$$\lim_{t\uparrow\infty} \frac{\sigma^2\big(\bar{X}_0^d(t)\big)}{L(t)t^{2H}} = \frac{\mu^2}{(2H-1)H(H-1)}. \tag{63}$$

Next, we assume that the random lifetime T^1 defined above is itself infinitely divisible and more specifically that T^1 is in the compound exponential class (see [20] for motivations). By doing so, we define a class of doubly compound exponential stationary renewal processes.

Corollary 11. *Suppose $\phi_{T^1}(p) = 1/(1+\varphi(p))$ for some Laplace exponent $\varphi(p)$. Then,*

(i)

$$\Phi^\wedge_{\bar{X}_0^d}(p,\lambda) = \frac{1}{p}\left[1 - \frac{1}{p}\frac{1-\phi_Z(\lambda)}{1+\varphi(p)^{-1}(1-\phi_Z(\lambda))}\right] \tag{64}$$

$$= \frac{1}{p}\left[1 - \frac{1}{p}\frac{\varphi(p)\psi(\lambda)}{\varphi(p)+\psi(\lambda)+\varphi(p)\psi(\lambda)}\right]. \tag{65}$$

(ii) Define $\bar{T}_0^d(x) = \inf\big(t \geqslant 0 : \bar{X}_0^d(t) > x\big)$ the generalized inverse of $\bar{X}_0^d(\cdot)$. Then, with $\Phi^\wedge_{\bar{T}_0^d}(\lambda, p) := \int_0^\infty \mathrm{d}x\, \mathrm{e}^{-\lambda x}\boldsymbol{E}\,\mathrm{e}^{-p\bar{T}_0^d(x)}$, we have

$$\Phi^\wedge_{\bar{T}_0^d}(\lambda, p) = \frac{1}{\lambda}\left[\frac{(1-\phi_{T^1}(p))/p}{1+\psi(\lambda)^{-1}(1-\phi_{T^1}(p))}\right] = \frac{1}{\lambda p}\frac{\psi(\lambda)\varphi(p)}{\psi(\lambda)+\varphi(p)+\psi(\lambda)\varphi(p)}. \tag{66}$$

Proof.

(i) Recalling $v^\wedge(p) = 1/(1-\phi_{T^1}(p))$, we get with this particular $\phi_{T^1}(p)$: $v^\wedge(p) = (1+\varphi(p))/\varphi(p)$. Putting this expression in (61) and recalling that $v_+^\wedge(p) = \phi_{T^1}(p)/(1-\phi_{T^1}(p)) = \varphi(p)^{-1}$, (62) yields the first part of (i). The second part is elementary algebra.

(ii) Clearly, the events $\bar{X}_0^d(t) > x$ and $\bar{T}_0^d(x) \leqslant t$ coincide. As a result, applying Fubini's formula yields $\Phi^\wedge_{\bar{T}_0^d}(\lambda, p) = \frac{1}{\lambda}\big[1 - p\Phi^\wedge_{\bar{X}_0^d}(p,\lambda)\big]$. Observing that (i) may be written as

$$\Phi^\wedge_{\bar{X}_0^d}(p,\lambda) = \frac{1}{p}\left[1 - \frac{1}{p}\frac{1-\phi_{T^1}(p)}{1+\psi(\lambda)^{-1}(1-\phi_{T^1}(p))}\right],$$

we get

$$\Phi^\wedge_{\bar{T}_0^d}(\lambda, p) = \frac{1}{\lambda p}\left[\frac{1-\phi_{T^1}(p)}{1+\psi(\lambda)^{-1}(1-\phi_{T^1}(p))}\right].$$

We conclude from (66) and (44) that $\bar{T}_0^d(x)$ is a pure compound renewal process whose initial jump at $x = 0$ has LST $(1 - \phi_{T^1}(p))/p$. □

It is known that $\Phi_{\bar{T}(\tau)}(p) = \mathrm{e}^{-\tau\varphi(p)}$ is the LST of some Lévy subordinator $\bar{T}(\tau)$, $\tau \geqslant 0$. Next consider the average time spent by $\bar{T}(\tau)$ below $t > 0$ and set

$$V_+(t) := \boldsymbol{E}\int_0^\infty \mathbf{1}(\bar{T}(\tau) \leqslant t)\,\mathrm{d}\tau. \tag{67}$$

This function is known as the potential function of $\bar{T}(\cdot)$.

Note that $\int_0^\infty \mathbf{1}(\bar{T}(\tau) \leqslant t)\,\mathrm{d}\tau = \inf(\tau : \bar{T}(\tau) > t) := \bar{\mathcal{T}}(t)$.

Corollary 12. *This potential function admits the alternative representation*

$$V_+(t) = \boldsymbol{E}\left(\sum_{n\geqslant 1} \mathbf{1}(\bar{T}^1(n) \leqslant t)\right) \tag{68}$$

where $\bar{T}^1(n) := \sum_{m=1}^n T_m^1$, *with* $(T^1; T_m^1, m \geqslant 1)$ *is some iid sequence characterized by the LST:* $\phi_{T^1}(p) := \boldsymbol{E}\,\mathrm{e}^{-pT^1} = 1/(1 + \varphi(p))$.

Proof. The LST, say $v_+^\wedge(p)$, of the potential measure associated with V_+ in (58) is easily shown to be

$$v_+^\wedge(p) = \frac{1}{\varphi(p)} \qquad p > 0$$

which, for some LST ϕ_{T^1}, is also

$$v_+^\wedge(p) = \frac{1}{\varphi(p)} = \frac{\phi_{T^1}(p)}{1 - \phi_{T^1}(p)}.$$

From (7), (12), this means that (68) holds. □

3.2. Lévy renewal processes and their inverse

We now proceed with yet another class of limiting renewal processes which we shall call Lévy renewal processes as they are to compound renewal processes what Lévy subordinators are to compound Poisson processes.

Let $T^1 \geqslant 0$ be the proper waiting time of some pure renewal process. Assume $\boldsymbol{E}(T^1) = 1$ and that T^1 is in the compound exponential class. Thus $\phi_{T^1}(p) = 1/(1 + \varphi(p))$ for some Laplace exponent $\varphi(p)$ with $\varphi^{(1)}(0) = 1$. Let $\varepsilon > 0$ and $\theta_\varepsilon \in (0, 1)$, with $\theta_\varepsilon \downarrow 0$ as $\varepsilon \downarrow 0$. Consider a waiting time T_ε whose LST is given by $\phi_{T_\varepsilon}(p) = \phi_{T^1}(p)^{\theta_\varepsilon}$. Clearly $\boldsymbol{E}(T_\varepsilon) = \theta_\varepsilon$ and the renewal process generated by such T_ε has renewal measure characterized by

$$u_+^\wedge(p) = \frac{(1 + \varphi(p))^{-\theta_\varepsilon}}{1 - (1 + \varphi(p))^{-\theta_\varepsilon}} \sim_{\theta_\varepsilon \downarrow 0} \frac{1}{\theta_\varepsilon \varphi(p)}. \tag{69}$$

In (56), suppose the support of X is $[\varepsilon, \infty]$, with $\varepsilon > 0$. Call this random variable X_ε. With $\bar{\Pi}_Z(x) := \Pi_Z[x, \infty)$, the tail of the Lévy measure, suppose the cpdf of X_ε reads $\bar{F}_{X_\varepsilon}(x) = \bar{\Pi}_Z(x \vee \varepsilon)/\bar{\Pi}_Z(\varepsilon)$. Call the delayed compound renewal processes with such a pair T_ε, X_ε $\bar{X}_\varepsilon^d(t)$.

We shall now pass to the limit $\varepsilon \downarrow 0$, while $\theta_\varepsilon \bar{\Pi}_Z(\varepsilon) = 1$ in (56). Note that in this process $\bar{\Pi}_Z(\varepsilon) \to \infty$ and that $\theta_\varepsilon \to 0$ (the average time between consecutive jumps tends to 0, while the probability mass of the jumps X_ε concentrates at 0). It turns out that $\bar{X}_\varepsilon^d(\cdot)$ converges weakly to some limiting process, say $\bar{X}_L(\cdot)$, which we shall call Lévy stationary renewal processes. Indeed, we obtain

Theorem 13. *The following weak convergence holds*

$$\left(\bar{X}^d_\varepsilon(t), t \geqslant 0\right) \xrightarrow{d} (\bar{X}_L(t), t \geqslant 0) \qquad \varepsilon \downarrow 0 \tag{70}$$

where $\bar{X}_L(\cdot)$ *is a Lévy renewal process with LST functional*

$$\Phi^{\wedge}_{\bar{X}_L}(p, \lambda) = \frac{1}{p}\left[1 - \frac{1}{p}\frac{\psi(\lambda)}{1 + \varphi(p)^{-1}\psi(\lambda)}\right]. \tag{71}$$

Proof. Observing as before that $\frac{1}{\theta_\varepsilon}\left(1 - \phi_{X_\varepsilon}(\lambda)\right) \to_{\varepsilon\downarrow 0} \psi(\lambda)$ and using (69), we obtain from (56) the announced LST functional in (71). Note, from (71), that the unidimensional distribution can be characterized by the LST

$$\boldsymbol{E}\,\mathrm{e}^{-\lambda \bar{X}_L(t)} = \mathbf{1}(t \geqslant 0) + \sum_{n\geqslant 1}(-1)^n \psi(\lambda)^n \int_0^t V_+^{*(n-1)}(s)\,\mathrm{d}s \tag{72}$$

with $V_+^{*n}(s)$ the nth convolution of the temporal renewal function $V_+(s)$ defined by: $\int_0^\infty \mathrm{e}^{-ps}\,\mathrm{d}V_+(s) = \varphi(p)^{-1}$. □

From (72), if $\psi^{(1)}(0) < \infty$ we have $\boldsymbol{E}(\bar{X}_L(t)) = \psi^{(1)}(0)t$, for all times. The variance function therefore is $\sigma^2(\bar{X}_L(t)) = \boldsymbol{E}(\bar{X}_L(t))^2 - (\psi^{(1)}(0)t)^2$ and, from (55), (56), we obtain similarly

Corollary 14. *Assume* $-\psi^{(2)}(0) < \infty$. *Under* (T_δ) *for* T^1, *the long-range dependence behaviour (47) also holds for the limiting Lévy process and, with* $H = (2-\delta)/2 \in (1/2, 1)$

$$\lim_{t\uparrow\infty} \frac{\sigma^2\left(\bar{X}^d_L(t)\right)}{L(t)t^{2H}} = \frac{\psi^{(1)}(0)^2}{(2H-1)H(H-1)}. \tag{73}$$

Remark 3. Assume $\phi_{T^1}(p) = 1/(1+p)$ which means that T^1 has exponential distribution with mean 1. Then, $\varphi(p) = p$ and from (71), $\Phi^{\wedge}_{\bar{X}_L}(p, \lambda) = \frac{1}{p+\psi(\lambda)}$ which means $\Phi_{\bar{X}_L}(t, \lambda) = \exp[-t\psi(\lambda)]$ which is the Lévy–Khintchine representation of subordinators with sii and Laplace exponent $\psi(\lambda)$. This can also be seen from (72) because, in this case, $V_+^{*(n-1)}(s) = s^{n-1}/(n-1)!$.

Remark 4. Let $\bar{T}_L(x) := \inf(t \geqslant 0 : \bar{X}_L(t) > x)$ be the generalized inverse process of $\bar{X}_L(\cdot)$. Again, the events $\bar{X}_L(t) > x$ and $\bar{T}_L(x) \leqslant t$ coincide. With $\Phi^{\wedge}_{\bar{T}_L}(\lambda, p) := \int_0^\infty \mathrm{d}x\,\mathrm{e}^{-\lambda x}\boldsymbol{E}\,\mathrm{e}^{-p\bar{T}_L(x)}$, the LST functional of $\bar{T}_L(\cdot)$, applying Fubini's theorem, we get $\Phi^{\wedge}_{\bar{T}_L}(\lambda, p) = \frac{1}{\lambda}\left(1 - p\Phi^{\wedge}_{\bar{X}_L}(p, \lambda)\right)$. As a result, from (71)

$$\Phi^{\wedge}_{\bar{T}_L}(\lambda, p) = \frac{1}{\lambda}\left[\frac{\varphi(p)/p}{1 + \psi(\lambda)^{-1}\varphi(p)}\right]. \tag{74}$$

Alternatively,

$$\boldsymbol{E}\,\mathrm{e}^{-p\bar{T}_L(x)} = \frac{1}{p}\sum_{n\geqslant 0}(-1)^n \varphi(p)^{n+1}\mathcal{U}_+^{*n}(x)$$

with $\mathcal{U}_+^{*n}(x)$ the nth convolution of the spatial renewal function $\mathcal{U}_+(x)$ defined by: $\int_0^\infty \mathrm{e}^{-\lambda x}\,\mathrm{d}\mathcal{U}_+(x) = \psi(\lambda)^{-1}$.

Note that if $\bar{X}_L(\cdot)$ is a subordinator with sii, then $\varphi(p) = p$ and $\Phi^{\wedge}_{\bar{T}_L}(\lambda, p) = \frac{1}{\lambda}\left[\frac{1}{1+\psi(\lambda)^{-1}p}\right]$, showing that for inverse subordinators

$$\boldsymbol{E}\,\mathrm{e}^{-p\bar{T}_L(x)} = \sum_{n\geqslant 0}(-p)^n \mathcal{U}_+^{*n}(x).$$

Remark 5. Consider the partial sums $\bar{T}(n) = \sum_{m=1}^{n} T_m$, with $(T; T_m, m \geqslant 1)$ an iid sample where T is *ID* with mean θ and $\phi_T(p) = (1 + \varphi(p))^{-\theta}$, as before. Consider also the subordinator with sii $(\bar{T}(\tau), \tau \geqslant 0)$ as the one whose LST reads $\phi_{T(\tau)}(p) = \exp\{-\tau\varphi(p)\}, \tau \geqslant 0$. With $[x]$ standing for the integral part of $x > 0$, it may then easily be checked that the following convergence in distribution holds, as $\theta \downarrow 0^+$

$$\left(\bar{T}_{[\tau/\theta]}, \tau \geqslant 0\right) \xrightarrow{d} (\bar{T}(\tau), \tau \geqslant 0). \tag{75}$$

Let now $\bar{\mathcal{T}}(t) := \inf(t > 0 : \bar{T}(\tau) > t)$ be the first passage time at level t of $\bar{T}(\tau)$. Define the overshoot and undershoot at t respectively as

$$A_L(t) = \bar{T}(\bar{\mathcal{T}}(t)) - t \qquad B_L(t) = t - \bar{T}(\bar{\mathcal{T}}(t)_-). \tag{76}$$

Let

$$S_L(t) = A_L(t) + B_L(t) = \bar{T}(\bar{\mathcal{T}}(t)) - \bar{T}(\bar{\mathcal{T}}(t)_-) \tag{77}$$

be the jump's height of $\bar{T}(\cdot)$ which includes t. Finally, let $\bar{A}_L(t) := A_L(t) + t = \bar{T}(\bar{\mathcal{T}}(t))$. Just as in example 1, its LST functional can be obtained similarly while passing to the limit $\theta \downarrow 0$. With $\Phi^{\wedge}_{\bar{A}_L}(p, \lambda) := \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-pt} \boldsymbol{E}\, \mathrm{e}^{-\lambda \bar{A}_L(t)}$, we find

$$\Phi^{\wedge}_{\bar{A}_L}(p, \lambda) = \frac{1}{p}\left[1 - \frac{\varphi(\lambda)}{\varphi(p + \lambda)}\right]. \tag{78}$$

Observing that $\Phi^{\wedge}_{A_L}(p, \lambda) = \Phi^{\wedge}_{\bar{A}_L}(p - \lambda, \lambda)$, we obtain $\Phi^{\wedge}_{A_L}(p, \lambda) = \frac{1}{p-\lambda}\left[1 - \frac{\varphi(\lambda)}{\varphi(p)}\right]$. This shows that

$$\lim_{p \downarrow 0} p\Phi^{\wedge}_{A_L}(p, \lambda) = \varphi(\lambda)/\lambda.$$

In other words, the overshoot process $A_L(t)$ converges in distribution to a limiting overshoot variable, say A_L, whose LST is $\varphi(\lambda)/\lambda$. Stated differently, the pdf of A_L is

$$F_{A_L}(\tau) = \int_0^\tau \bar{\Pi}(t)\, \mathrm{d}t$$

where $\bar{\Pi}(t)$ is the tail of the Lévy measure $\Pi(\mathrm{d}t)$ defined by $\varphi(p) = \int_0^\infty (1 - \mathrm{e}^{-pt})\Pi(\mathrm{d}t)$, with $\int_0^\infty t\Pi(\mathrm{d}t) = 1$.

Note that, defining the inverse process $\bar{\mathcal{T}}_L(\tau) := \inf(t : \bar{A}_L(t) > \tau)$ and letting $\Phi^{\wedge}_{\bar{\mathcal{T}}_L}(\lambda, p) := \int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-\lambda\tau} \boldsymbol{E}\, \mathrm{e}^{-p\bar{\mathcal{T}}_L(\tau)}$, we have

$$\Phi^{\wedge}_{\bar{\mathcal{T}}_L}(\lambda, p) = \frac{1}{\lambda} \frac{\varphi(\lambda)}{\varphi(p + \lambda)}. \tag{79}$$

showing that these processes do not coincide.

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